# Design of Input-Output Observers for a Population of Systems with Bounded Frequency-Domain Variation using $\mathcal{DK}$-iteration

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May 27, 2026

# State Observers

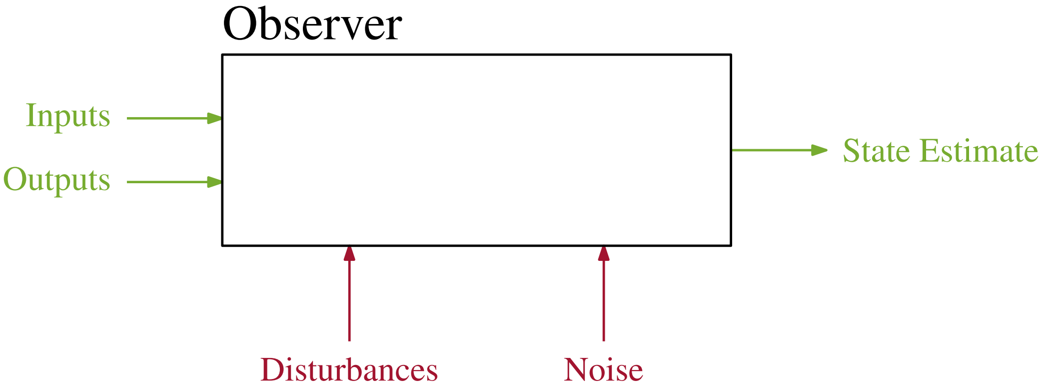


Figure 1: State observer diagram.

# State Observers

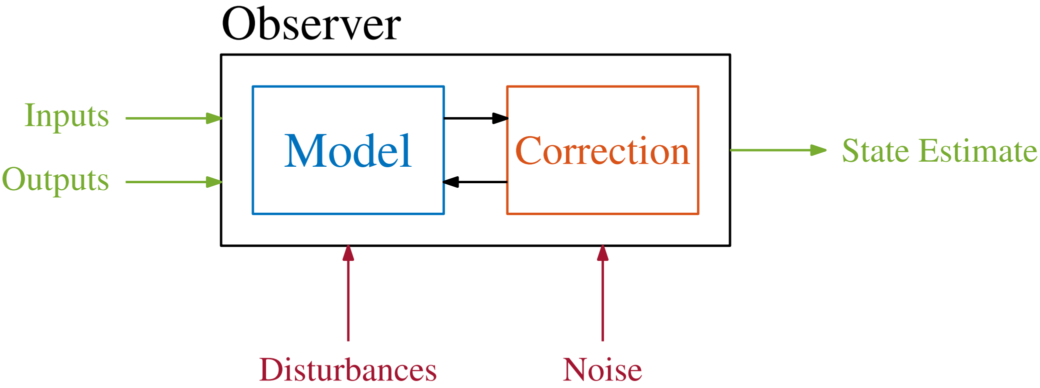


Figure 1: State observer diagram.

# State Observer Design

## Observer Design Procedure

**System $\rightarrow$ Model**

- First-principles modelling
- System identification

**Model $\rightarrow$ Correction Mechanism**

- Kalman Filter [1], [2]
- $\mathcal{H}_2/\mathcal{H}_\infty$ Observer [2], [3]
- Input-Output Observer [4], [5]

***Note: 1 Model $\rightarrow$ 1 Correction Mechanism***

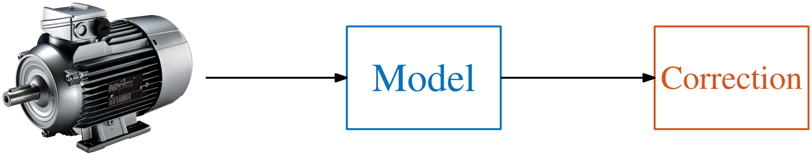


Figure 2: State observer design procedure.

# State Observer Design for a Population of Systems

## Problem Statement

**Population of Devices with Variations**
- Manufacturing
- Degradation
- Operating conditions

**Objective**
- Design an observer for each system that exploits a model of the particular device's dynamics.

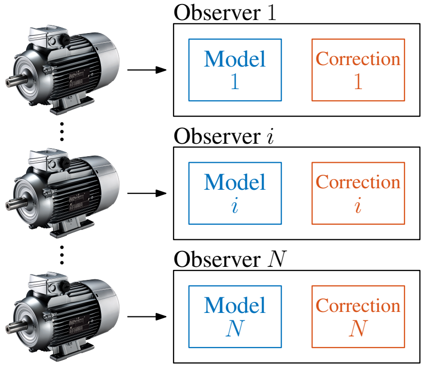


Figure 3: Observer design for a population of devices with variations and known models of their system dynamics.

# Observer Population Design Deficiencies

## Tailored Approach

- **Pro**: Good estimation accuracy across population.
- **Con**: Does not scale with population size.

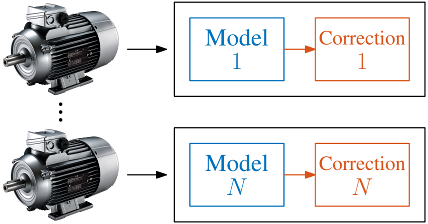


Figure 4: Tailored observer design approach for a population of systems.

# Observer Population Design Deficiencies

## Nominal Approach

- **Pro**: Scales with population size.
- **Con**: No estimation accuracy guarantees across population.

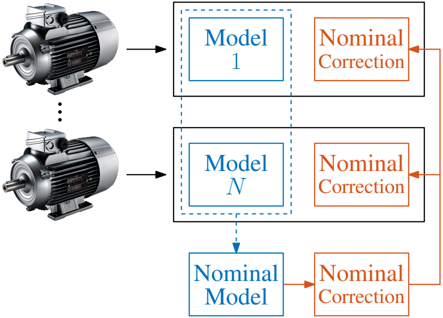


Figure 5: Nominal observer design approach for a population of systems.

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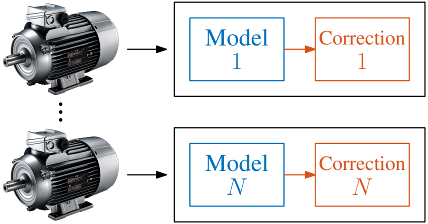


Figure 4: Tailored observer design approach for a population of systems.

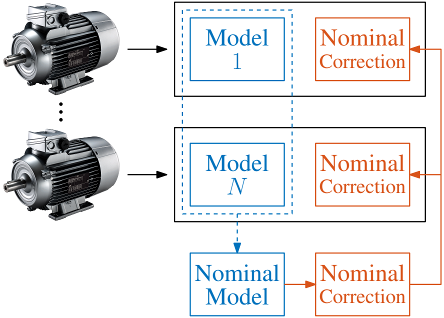


Figure 5: Nominal observer design approach for a population of systems.

# Proposed Observer Population Design Methodology

## Proposed Approach

- **Method**:

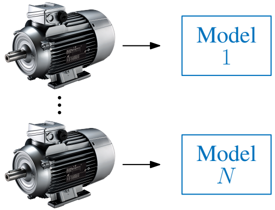


Figure 6: Robust observer design approach for a population of systems.

# Proposed Observer Population Design Methodology

## Proposed Approach

- **Method**:
  - Compute *nominal model* and *variation characterization* across population.

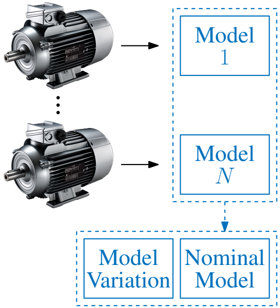


Figure 6: Robust observer design approach for a population of systems.

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## Proposed Approach

- **Method**:
  - Compute *nominal model* and *variation characterization* across population.
  - Synthesize *robust correction mechanism* using robust control theory.

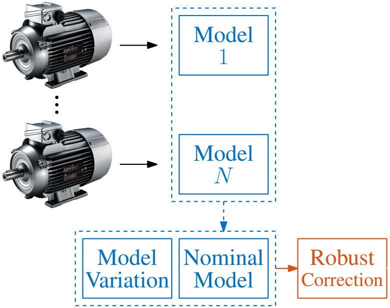


Figure 6: Robust observer design approach for a population of systems.

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  - Compute *nominal model* and *variation characterization* across population.
  - Synthesize *robust correction mechanism* using robust control theory.
  - Form observers with robust correction mechanism and particular system model.

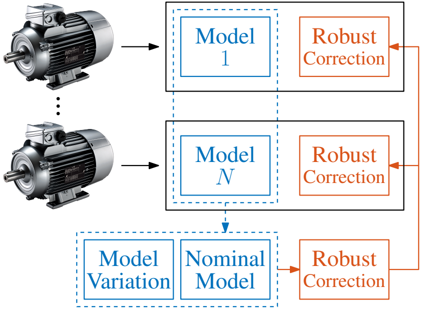


Figure 6: Robust observer design approach for a population of systems.

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- **Method**:
  - Compute *nominal model* and *variation characterization* across population.
  - Synthesize *robust correction mechanism* using robust control theory.
  - Form observers with robust correction mechanism and particular system model.
- **Pros**:
  - Estimation accuracy guarantees across population.
  - Scales with population size.

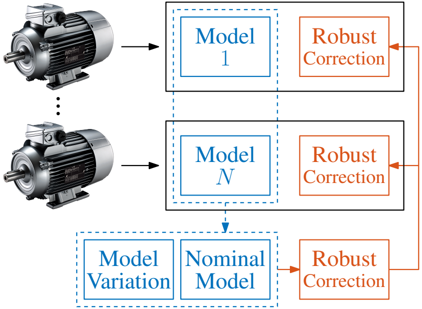


Figure 6: Robust observer design approach for a population of systems.

# Variation Characterization

## Unstructured LTI Uncertainty [6]

- Inverse Multiplicative Input Uncertainty Model

$$\mathbf{G}(s) = \mathbf{G}_0(s)(\mathbf{1} - \mathbf{W}_\triangle(s)\mathbf{\Delta}(s))^{-1}, \tag{1}$$

- Norm-bounded LTI Uncertainty

$$\|\mathbf{\Delta}(s)\|_\infty = \sup_\omega \bar{\sigma}(\mathbf{\Delta}(j\omega)) \leq 1 \tag{2}$$

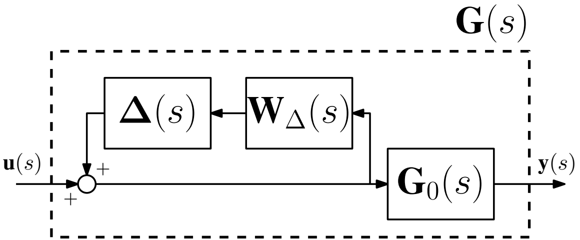


Figure 7: Inverse multiplicative input uncertainty representation of population of systems.

# Input-Output Observer [4], [5]

## System Model

**Systems**

- Process model $\mathbf{G}(s) = (s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}$
  - Varies across population
- Measurement model $\mathbf{C}$
  - Constant across population

**Signals**

- Input $\mathbf{u}(s)$
- State $\mathbf{x}(s)$
- Output $\mathbf{y}(s)$

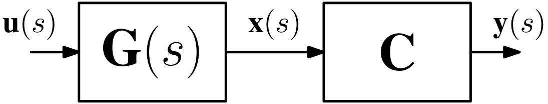


Figure 8: Process model and measurement model.

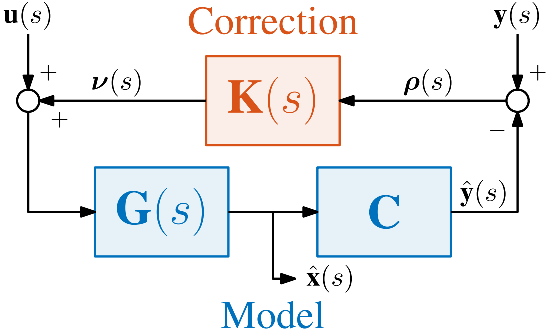


Figure 9: Input-output observer.

# Input-Output Observer Error Dynamics

## Observer Error Dynamics

- Disturbances $\rightarrow$ Estimate errors
  - $\mathbf{d}_x(s), \mathbf{d}_u(s), \mathbf{n}(s) \rightarrow \mathbf{e}_x(s), \mathbf{e}_y(s)$
- Standard  control problem

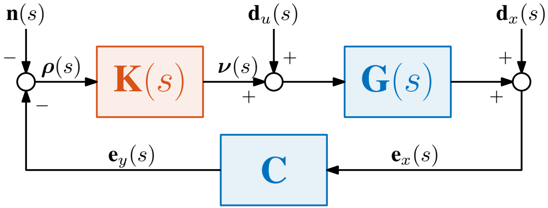


Figure 10: Input-output observer error dynamics.

# Input-Output Observer Error Dynamics

## Observer Error Dynamics

- Disturbances $\rightarrow$ Estimate errors
  - $\mathbf{d}_x(s), \mathbf{d}_u(s), \mathbf{n}(s) \rightarrow \mathbf{e}_x(s), \mathbf{e}_y(s)$
- Standard *robust* control problem

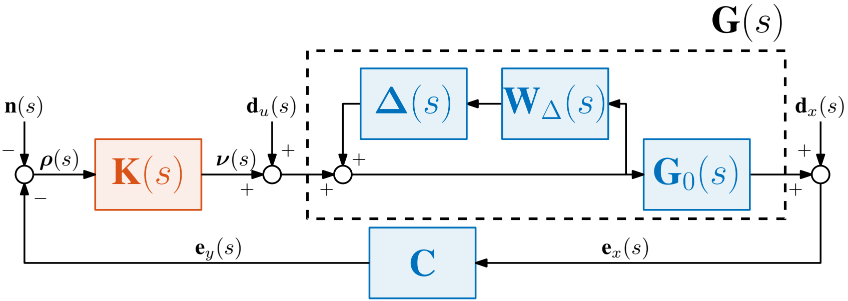


Figure 10: Input-output observer error dynamics.

# Robust Control Theory

## Robust Control Problem

- Generalized plant $\mathbf{P}(s)$
- Correction filter $\mathbf{K}(s)$
- Normalized perturbation $\mathbf{\Delta}(s)$

## Robust Performance Problem [6]

- Robust Performance Condition

$$\|\mathbf{w}(s) \mapsto \mathbf{z}(s)\|_\infty \leq 1 \quad \forall \|\mathbf{\Delta}(s)\|_\infty \leq 1 \tag{3}$$

- Structured Singular Value (SSV) Condition

$$\mu_{\hat{\mathbf{\Delta}}}(\omega) \leq 1 \quad \forall \omega \in [0, +\infty) \tag{4}$$

- Robust Controller Synthesis
  - *DK*-iteration [6]

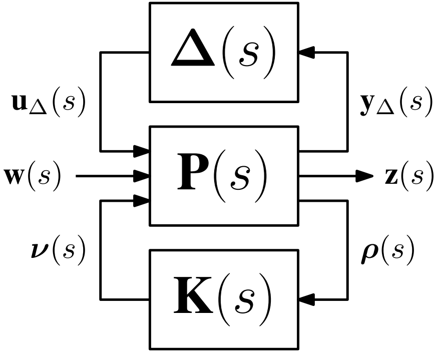


Figure 11: General robust control problem.

# State Estimation Experiment: Flexible Joint Robot

## Estimation Objective

**Estimated State**

- Motor positions (shoulder $\theta_1$ and elbow $\theta_2$)
- Flexible link deflections (shoulder $\alpha_1$ and elbow $\alpha_2$)

**Measured Output**

- Motor positions (shoulder $\theta_1$ and elbow $\theta_2$)

**Measured Inputs**

- Motor currents (shoulder $i_1$ and elbow $i_2$)

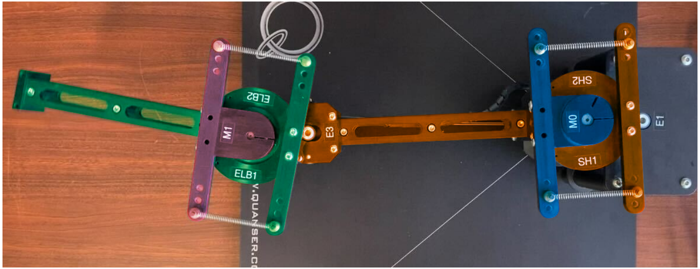


Figure 12: Quanser *2 DOF Serial Flexible Joint* with colorized joints.

# Population System Identification

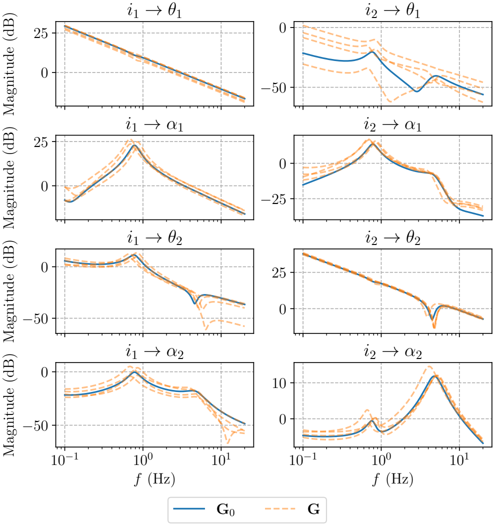


Figure 13: Magnitude response of flexible joint robot with different joint stiffness configurations.

# Variation Characterization

## Uncertainty Model Fit

1. Compute $\mathbf{E}_{iI}^{[k]}(j\omega)$ via
$$\mathbf{G}^{[k]}(j\omega)\mathbf{E}_{iI}^{[k]}(j\omega) = \mathbf{G}^{[k]}(j\omega) - \mathbf{G}_0(j\omega) \tag{5}$$
2. Compute
$$\max_k \bar{\sigma}(\mathbf{E}_{iI}^{[k]}(j\omega)) \tag{6}$$
3. Fit an overbounding stable and minimum-phase $\mathbf{W}_\Delta(s) = w_\Delta(s)\mathbf{1}$

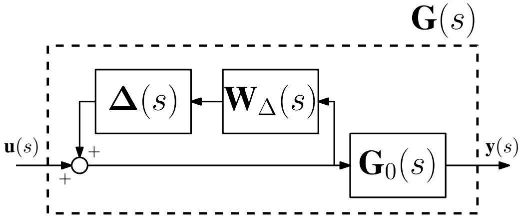


Figure 13: Inverse multiplicative input uncertainty.

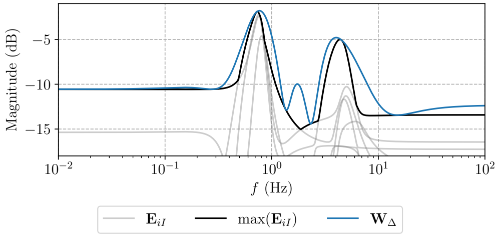


Figure 14: Singular value response of uncertainty residuals and uncertainty weight.

# Generalized Robust Control Problem

## Generalized Control

**Exogenous Disturbances $\mathbf{w}(s)$**
- Input disturbance $\mathbf{d}_u(s)$
- Sensor noise $\mathbf{n}(s)$

**Performance Outputs $\mathbf{z}(s)$**
- Filtered estimate error $\mathbf{z}_1(s)$
- Filtered input correction $\mathbf{z}_2(s)$

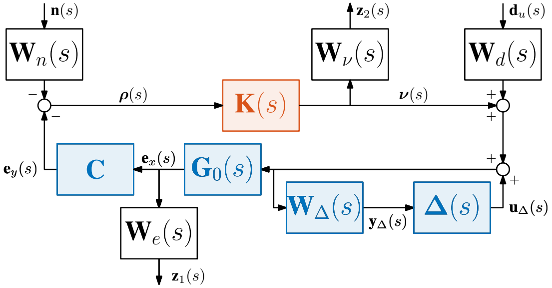


Figure 15: Input-output observer error dynamics augmented with LTI weighting filters.

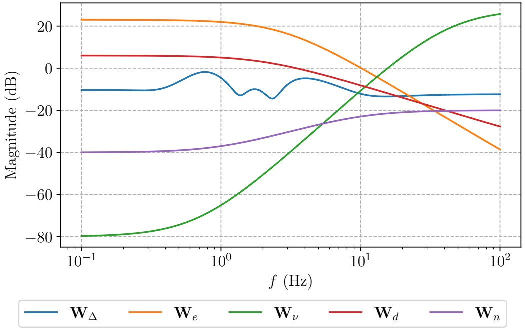


Figure 16: Singular value response of the generalized control problem weights.

# Results: $DK$-iteration

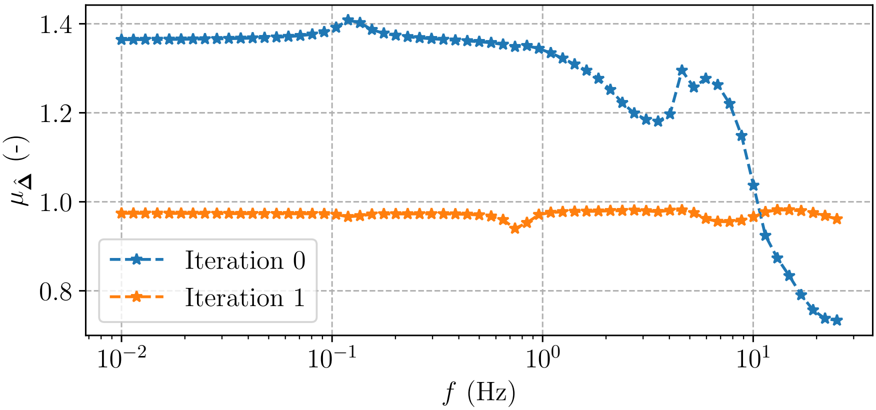


Figure 17: Structured singular value response across iterations of $DK$-iteration.

# Results: Estimation Performance

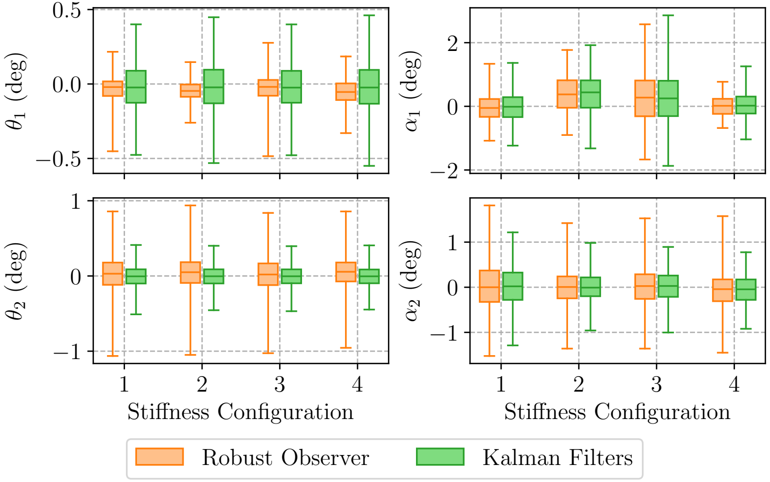


Figure 18: Absolute estimation error box plots across stiffness configurations for a validation dataset.

# Summary

- Proposed a robust observer synthesis methodology for a population of systems with variation in their system dynamics.
  1. Extract nominal model and uncertainty characterization using LTI norm-bounded uncertainty.
  2. Synthesize a robust input-output observer correction filter using $DK$-iteration.
  3. Construct the input-output observers for the population of system by pairing the system model and correction filter.
- Demonstrate the observer synthesis methodology on a population of flexible joint robots with different joint stiffnesses.

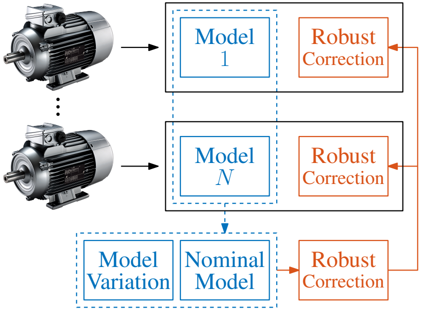


Figure 6: Robust observer design approach for a population of systems.

# Questions?

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Paper

Code

# References


[1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*. Prentice-Hall Information and System Sciences Series, 1979, 370 pp.

[2] D. Simon, *Optimal State Estimation: Kalman, $\mathcal{H}_\infty$ and Nonlinear Approaches*. Hoboken, N.J: Wiley-Interscience, 2006, 526 pp.

[3] M. Green and D. J. N. Limebeer, *Linear Robust Control*. Courier Corporation, Sep. 19, 2012, 562 pp.

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[5] H. J. Marquez and M. Riaz, "Robust state observer design with application to an industrial boiler system," *Control Engineering Practice*, vol. 13, no. 6, pp. 713–728, Jun. 1, 2005.

[6] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design*, 2 edition. Chichester: Wiley-Interscience, Nov. 4, 2005, 592 pp.